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COMPETITIVE OUTCOMES IN THE CORES OF MARKET GAMES

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## ABSTRACT

The competitive outcomes of an economic system are known, under quite general conditions, always to lie in the core of the associated cooperative game. It is shown here that every "market game" (i.e., one that arises from an exchange economy with money) can be represented by a "direct market" whose competitive outcomes completely fill the core. It is also shown that it can be represented by a market having any given core outcome as its unique competitive outcome, or, more generally, having any given compact convex subset of the core as its full set of competitive outcomes.

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# COMPETITIVE OUTCOMES IN THE CORES OF MARKET GAMES[1]

by

L. S. Shapley[2] and M. Shubik[3]

## 1. INTRODUCTION

In a previous paper[4], the authors introduced a class of cooperative n-person games in characteristic-function form, called "market games," which come from trading economies in which the traders measure utility in money. This class of games was shown to coincide with the class of "totally balanced" games, i.e., games that have nonempty cores and all of whose subgames have nonempty cores as well. In this paper we shall compare the cores of market games with the competitive equilibria of the markets that they come from. We first consider the "direct market" of a market game, and discover that its competitive outcomes fill the entire core (Theorem 1). We then take an arbitrary point in the core and construct a market (actually a class of markets) which generates the given market game and which has the given core point as its only competitive outcome (Theorem 2). A modification of this construction yields an arbitrary closed convex subset of the core as the set of competitive outcomes.

Extensions of these results to games and markets without money (i.e., without transferable utility) will be considered in a subsequent paper.

## 2.  GAMES AND MARKETS

The reader is referred to Shapley and Shubik [1969] for a more extended discussion of the matters reviewed in this section.

A game, for our present purpose, will consist of a finite set N and a real-valued set function v, the latter defined on all the subsets of N and satisfying $v(\emptyset) = 0$. In the standard interpretation, N represents the set of players, and $v(S)$ represents the "worth" of S, i.e., the total amount of utility that the members of S can secure if they form a coalition and play the game without help from the other players.

Outcomes of the game are expressed as N-tuples of utility: $\alpha = \{\alpha^i : i \in N\}$, measured in some common monetary unit and called payoff vectors. A payoff vector $\alpha$ is said to be "feasible" if $\Sigma_{i \in N} \alpha^i \leq v(N)$; "efficient" if $\Sigma_{i \in N} \alpha^i = v(N)$; "individually rational" if $\alpha^i \geq v(\{i\})$ for each $i \in N$; and "coalitionally rational" if $\Sigma_{i \in S} \alpha^i \geq v(S)$ for each $S \subseteq N$. The set of feasible, coalitionally rational payoff vectors is called the core of the game; thus, $\alpha$ is the core if and only if

(1)         $\alpha \cdot e^S \geq v(S)$, all $S \subset N$, and $\alpha \cdot e^N = v(N)$,

where $e^S$ denotes the N-vector having $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \in N - S$. Geometrically, the core is a compact convex polyhedron, possibly empty.

It is well known[5] that nonemptiness of the core is equivalent to the game being balanced, in the sense that

$$(2) \qquad \sum_{S \subseteq N} \gamma_S v(S) \leqq v(N)$$

holds for every set of weights $\gamma_S \geqq 0$, $S \subseteq N$ such that

$$(3) \qquad \sum_{S \subseteq N} \gamma_S e^S = e^N.$$

A balanced game v is said to be <u>totally balanced</u> if all of its sub-games, obtained by restricting v to the subsets of R, $R \subset N$, are also balanced.

A <u>market</u>, for our present purpose, will consist of a finite set N of "traders"; a finite set M of "commodities"; an $|M|$-dimensional euclidean orthant $E_+^M$ of "bundles"; and, for each $i \in N$, an <u>initial</u> <u>bundle</u> $a^i \in E_+^M$ and a continuous concave <u>utility function</u> $u^i$ from $E_+^M$ to the reals. In the interpretation,[6] utility is understood to be measured in units of money, and the traders may not only exchange the m commodities as initially supplied to them, but they may also transfer money in any amount. The <u>final payoff</u>, which the traders seek to maximize, is therefore found for each trader by adding his net gain of money to the utility of his final bundle of commodities.

A market generates a game in a natural way. First, for any $\emptyset \subset S \subseteq N$, let us define an S-<u>allocation</u> to be an indexed set $x^S = \{x^i : i \in S\}$ of bundles in $E_+^M$. An S-allocation is said to be <u>feasible</u> if

$$(4) \qquad \sum_{i \in S} x^i = \sum_{i \in S} a^i.$$

To generate the game, we set $v(\emptyset) = 0$ and, for each $\emptyset \subset S \subseteq N$ define

$$(5) \qquad v(S) = \max_{i \in S} \sum u^i(x^i),$$

letting the maximum run over all feasible S-allocations $x^S$. Any $v$ that can be represented in this way is called a market game.

By the "core" of a market we shall mean the core of the game it generates. Since market games are always balanced (indeed, the market games are precisely the totally balanced games), all markets have nonempty cores.

Obviously, many markets may generate the same game, while being dissimilar in other respects. In Shapley and Shubik [1969], a canonical representative for each such class of "game-theoretically equivalent" markets was introduced. We called it the direct market of the game, because of its simple form and the fact that the players themselves are, in a sense, the commodities being bought and sold.

To define the direct market for any game $v$, we first put the commodities M into one-to-one correspondence with the players N. The initial allocation is then given by $a^i = e^{\{i\}}$, $i \in N$. The traders all have "equal tastes," i.e., identical utility functions, $u^i \equiv u$, given by

$$(6) \qquad u(x) = \max_{S \subseteq N} \sum \gamma_S v(S),$$

the maximum running over all $\{\gamma_S \geq 0 : S \subseteq N\}$ satisfying [7]

(7) $$\sum_{S \subseteq N} \gamma_S e^S = x.$$

An intuitive explanation of this market can be given in terms of group activities or implicit production processes: associate with each coalition S an activity that "earns" the amount $v(S)$ if all members of S participate fully. For details, the reader is again referred to Shapley and Shubik [1969]. The technical justification for this construction, however, lies in the fact that if we start with a totally balanced set function v, then taking the market game of the direct market gives us v back again. On the other hand, if we start with a v that is not totally balanced, then we get back the so-called "cover" of v, which is the least totally balanced set function that is greater than or equal to v for all S.

From (6) and (7) and the definition of "balance" (i.e., (2), (3), relativized to R), one can verify that the utility function of the direct market of a totally balanced game satisfies

(8) $$u(e^R) = v(R), \qquad \text{all } R \subseteq N.$$

This shows that u may be regarded as an extension of the set function v to the domain of "fuzzy" sets, i.e., coalitions whose members may participate at fractional levels of intensity.[8]

## 3. THE COMPETITIVE EQUILIBRIUM

The so-called "competitive" solution is not a game theory con-
cept, but is based on the notion of an imposed schedule of prices
which, if accepted by all the members of the economy, will make it
possible to balance supply and demand in each commodity, "clearing
the market" to everyone's satisfaction. In our present setting, we
must remember that money (or "transferable utility") is implicitly
one of the commodities in exchange, so that the $i^{th}$ trader's complete
utility function has the form

$$(9) \qquad U^i(x^i, \xi^i) = u^i(x^i) + \xi^i.$$

Here $\xi^i$ denotes i's final money balance. If we wish to follow the
classical definition of "competition" we must keep this new commodity
explicitly in view. A typical price schedule can then be written
as

$$(10) \qquad (\pi, 1) = (\pi_1, \ldots, \pi_{|M|}, 1),$$

where we have set the price of money at 1, as a convenient normaliza-
tion. These prices serve to evaluate everything, including money,
in terms of some new accounting unit.

Acting "competitively" in the face of (10), the $i^{th}$ trader will
seek to maximize (9) subject to the "budget" constraint

$$(11) \qquad \pi \cdot x^i + \xi^i = \pi \cdot a^i + \xi_o^i,$$

which equates the inflow and outflow of the accounting unit. Here $\zeta_o^i$ denotes i's initial money balance. On the assumption of freely trans- ferable utility, $\xi^i$ is an unrestricted variable, so we may solve (11) and eliminate $\xi^i$ from (9). Trader i's goal can now be restated: to maximize

$$(12) \qquad u^i(x^i) + \zeta_o^i + \pi \cdot (a^i - x^i)$$

where $x^i$ is now chosen unrestrictedly from $E_+^M$. For the price schedule $(\pi, 1)$ to be in competitive equilibrium, there must exist a set of maximizing choices by the different traders that fit together to form a feasible N-allocation, since only then can the market be cleared to everyone's satisfaction.

By this roundabout path we have arrived at the desired defini- tion.[9] A <u>competitive solution</u> in our model is an ordered pair $(\pi, z^N)$, where $\pi$ is an arbitrary N-vector of prices[10] and $z^N$ is a feasible N-allocation, such that

$$(13) \qquad u^i(z^i) - \pi \cdot z^i = \max_{x^i \in E_+^M} [u^i(x^i) - \pi \cdot x^i], \qquad \text{all } i \in N.$$

In other words, each trader maximizes his "trading profit." Note that we have omitted the terms $\zeta_o^i$ and $\pi \cdot a^i$ appearing in (12), as they are irrelevant to the maximization problem.

Moving to the payoff space, we shall call a vector $\alpha$ <u>competitive</u> if it arises from a competitive solution $(\pi, z^N)$, thus:

$$(14) \qquad \alpha^i = u^i(z^i) - \pi \cdot (z^i - a^i), \qquad \text{all } i \in N.$$

It is not hard to establish that the competitive allocations are just those that maximize _total_ utility, $\Sigma_{i \in N} u^i(x^i)$. It follows that the set of all competitive payoff vectors of a market is nonempty, compact, and convex; moreover, it is a subset of the core. To show the latter, suppose that $\alpha$ is competitive but not in the core. Then $v(S) > \alpha \cdot e^S$ for some $S \subset N$. This means, by (5), that there would be a feasible S-allocation $x^S$ such that

$$\sum_{i \in S} u^i(x^i) > \sum_{i \in S} (u^i(z^i) - \pi \cdot (z^i - a^i)).$$

But $\Sigma_{i \in S} \pi \cdot a^i = \Sigma_{i \in S} \pi \cdot x^i$, by (4), and so we would have

$$\sum_{i \in S} (u^i(x^i) - \pi \cdot x^i) > \sum_{i \in S} (u^i(z^i) - \pi \cdot z^i).$$

This contradicts the maximization (13). Hence each competitive payoff vector is a core vector. The reverse is not generally true, however; indeed, the competitive solution is often unique whereas the core is typically a set of $|N| - 1$ dimensions.[11] The following theorem shows, however, that in a direct market the set of competitive vectors and the core coincide.

THEOREM 1. Every payoff vector in the core of a game is competitive in the direct market of that game.

Proof. Let $\alpha$ be in the core of $v$.[12] The idea will be to show that $\alpha$ itself can be used as a competitive price vector for the direct market of the game. By (8) and (1) we have

$$(15) \qquad\qquad u(e^N) = v(N) = \alpha \cdot e^N.$$

In other words, the "value" of the total supply of goods $e^N$ is the same in utility terms as it is when computed using $\alpha$ as a price vector.

Next, take an arbitrary bundle $x \in E_+^M$, and let $\{\gamma_S : S \subseteq N\}$ be any set of nonnegative coefficients satisfying (7). Then, by (1) and (7),

$$\sum_{S \subseteq N} \gamma_S v(S) \leq \sum_{S \subseteq N} \gamma_S (\alpha \cdot e^S) = \alpha \cdot \sum_{S \subseteq N} \gamma_S e^S = \alpha \cdot x.$$

Hence, by (6),

$$(16) \qquad\qquad u(x) \leq \alpha \cdot x.$$

We can now show that $\alpha$ is a competitive payoff vector. Define prices by $\pi_i = \alpha^i$, $i \in N$. At these prices, a trader trying to choose $x^i$ to maximize his "trading profit" $u(x^i) - \pi \cdot x^i$, as in (13), will find that he can't make it positive, because of (16), but that he can make it zero by choosing $x^i$ to be the bundle $e^N$, because of (15).[13] By the homogeneity of $u$, any fraction $f^i$ of that bundle also yields a trading profit of zero. So we can construct a competitive solution $(\pi, z^N)$ by taking $z^i = f^i e^N$, where the $f^i$ are any nonnegative numbers

that sum to $\lrcorner$. Moreover, $(\pi, z^N)$ yields the desired payoff vector $\alpha$, since we have (see (14))

$$u(z^i) - \pi.z^i + \pi.a^i = 0 + \pi_i = \alpha^i$$

for each $i \in N$. This completes the proof of Theorem 1.

Theorem 1 tells us that every point in the core of a market game is competitive for at least one of the associated markets--namely, the direct market. The next theorem refines this result, by showing that for each core point there are associated markets for which only that point is competitive.

> THEOREM 2. Among the markets that generate a given totally balanced game, there is at least one that has any given core point as its unique competitive payoff vector.

Proof. Let $v$ be totally balanced, let $d$ be a real number, and define the set function $v_d$ by

(17)      $v_d(S) = v(S)$, all $S \subset N$, and $v_d(N) = v(N) + d$.

The game $v_d$ is obviously totally balanced if $d \geq 0$. Let $u_d$ denote the utility function for the direct market of $v_d$ (see (6), (7)). By (8) we have

(18)      $u_d(e^S) = v_d(S)$,      all $S \subseteq N$.

Let $\alpha$ be an arbitrary core point of v, and define the function $u_{d,\alpha}$ by

$$(19) \qquad\qquad u_{d,\alpha}(x) = \min(u_d(x),\ \alpha \cdot x).$$

This is continuous and concave, and so it can serve as the utility function of a market with equal tastes, using the same commodity space and initial bundles as the direct market of v. We shall show that for any positive value of d this market has the properties claimed in the statement of the theorem, namely, (a) that its market game is v and (b) that its unique competitive payoff vector is $\alpha$.

(a) Since we have equal tastes and homogeneity of degree 1, the market game w generated by $u_{d,\alpha}$ is given by[14]

$$(20) \qquad\qquad w(S) = u_{d,\alpha}(\sum_{i \in S} a^i) = u_{d,\alpha}(e^S)$$

(compare (8)). By (19) and (18), this means that

$$w(S) = \min(v_d(S),\ \alpha \cdot e^S).$$

By (17) and (1), we see that this minimum is equal to v(S) both when $S \neq N$ and when $S = N$. Hence $w = v$ as claimed.

(b) As noted previously, the competitive solution maximizes total utility. Each competitive price vector will therefore be the gradient of a linear support to $u_{d,\alpha}(x)$ at $x = e^N$, since $u_{d,\alpha}$ is homogeneous of degree 1 and concave. But when d is positive, $u_{d,\alpha}(x)$

coincides with the linear function $\alpha \cdot x$ in at least a small neighborhood of $x = e^N$, by (19), because $u_d$ is continuous and

$$u_d(e^N) = v_d(N) > v(N) = \alpha \cdot e^N.$$

Hence $\alpha \cdot x$ is the only support at $x = e^N$, and so the unique competitive prices are $\pi_i = \alpha^i$, $i \in N$. As we saw in the proof of Theorem 1, these yield the payoff vector $\alpha$. This completes the proof of Theorem 2.

By a simple extension of this proof, a market can be constructed having any given closed convex subset of the core as the set of competitive payoff vectors. Indeed, it is only necessary to define

$$u_{d,A}(x) = \min_{\alpha \in A} u_{d,\alpha}(x),$$

where A is the desired convex set, and proceed as above. (We omit the details.) When A equals the core, the market obtained in this way is independent of the parameter d, and in fact reduces to the direct market; thus, this general construction unifies Theorems 1 and 2.

## 4. CONCLUDING REMARKS

The relation between the core and the competitive equilibrium can be viewed in terms of the information that is lost in passing from a market to the game that it generates. In the first place, all details concerning the commodities and their distribution among the traders are suppressed, since the analysis of the market game takes place in the utility or payoff space, not the allocation space. In the second place, the game actually takes cognizance of only a finite number of the possible outcomes of the market process, i.e., the best result for each coalition. Most of the detailed preference information contained in the utility functions is ignored, as may be seen clearly in Eqs. (8) or (20) above, where the utilities are evaluated only at the vertices of a cube.

This loss of information suggests that the core is a rather blunt solution concept, and accounts for the many-to-one correspondence between markets and their market games. It is not surprising that we were able to find plenty of markets (without even looking beyond the special type of markets where the commodities are identified with the traders) having just the competitive outcomes that we needed for our proofs.

In a subsequent note we shall consider the competitive equilibria of markets without money, using the framework set forth in the work of Billera and Bixby.[15] Here too, although the space of games is far richer (the analog of the function v being set-valued), there is a great loss of information and a similar many-one relationship between markets and games. It turns out, however, that the locus of

competitive payoffs is <u>not</u> the entire core of the game, but only a
certain "inner core." This inner core can also be characterized
game-theoretically, without explicit reference to any economic model.
It should be observed, however, that our method of analysis depends
heavily on the assumption of concave utility functions, which in turn
require cardinal utilities.[16] A radically new approach may be re-
quired before the analogous characterization of "ordinal" market games
and their competitive solutions can be obtained.

## FOOTNOTES

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[3] The Cowles Foundation, Yale University, New Haven, Connecticut, 06520.

[4] Shapley and Shubik [1969].

[5] See Shapley [1967].

[6] See Shapley and Shubik [1966], pp. 807-808.

[7] At least one such set of weights exists.  For example, take $\gamma_{\{i\}} = x_i$, $i \in N$; all other $\gamma_S = 0$.

[8] This extension from the vertices of the unit N-cube to all of $E_+^N$ is continuous, concave, and positively homogeneous of degree 1.  Such an extension exists only when v is totally balanced; see Shapley and Shubik [1969], Eq. 4-1, etc.  It may be contrasted with Owen's multi-linear extension [1972], which is always possible, and which is a more appropriate extension for studying the "value" solution concept. See also Aubin [1974] and Aumann and Shapley [1974], Ch. IV, esp. p. 166.

[9] We could, of course, have stated and justified this definition directly, keeping the role of transferable utility hidden beneath the notational surface.  We wished, however, to establish a firm connection with the standard definition of competitive equilibrium, and to clear up the "mystery" of the missing budget constraint in (13).

[10] We have not been assuming that utilities are nondecreasing, and so we do not assume here that prices are nonnegative.  This approach also entails using "=" rather than "$\leqq$" in the budget condition (11).  Note also that our prices $\pi_j$ are not just ratios, but have meaningful magnitudes.

[11] A sufficient condition for uniqueness of the competitive payoff is that the functions $u^i$ all be differentiable and that at least one competitive allocation be strictly positive.  A sufficient condition for the core to be full dimensional is that all the inequalities (2) (except the trivial case where $\gamma_N = 1$) be satisfied strictly.

[12] If v is not balanced the core is empty, so the theorem is vacuously true. If v is balanced but not totally balanced, the core coincides with that of the cover of v (see Shapley and Shubik [1969], Lemma 3), and hence with the core of the direct market of v.

[13] Indeed, if he could make it positive he could make it arbitrarily large, and no maximum would exist. But this is not true in general in our model, for with nonhomogeneous, concave utilities a positive trading profit is quite possible at competitive equilibrium.

[14] See Shapley and Shubik [1969], Eq. (3-3).

[15] See Billera and Bixby [1973, 1974] and Billera [1974].

[16] See Billera's comments [1974, pp 129-130].

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